

On upper bounds on the smallest size of a saturating set in a projective plane*

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Abstract

In a projective plane Π_q (not necessarily Desarguesian) of order q , a point subset S is saturating (or dense) if any point of $\Pi_q \setminus S$ is collinear with two points in S . Using probabilistic methods, the following upper bound on the smallest size $s(2, q)$ of a saturating set in Π_q is proved:

$$s(2, q) \leq 2\sqrt{(q+1)\ln(q+1)} + 2 \sim 2\sqrt{q\ln q}.$$

We also show that for any constant $c \geq 1$ a random point set of size k in Π_q with $2c\sqrt{(q+1)\ln(q+1)} + 2 \leq k < \frac{q^2-1}{q+2} \sim q$ is a saturating set with probability

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greater than $1 - 1/(q+1)^{2c^2-2}$. Our probabilistic approach is also applied to multiple saturating sets. A point set $S \subset \Pi_q$ is $(1, \mu)$ -saturating if for every point Q of $\Pi_q \setminus S$ the number of secants of S through Q is at least μ , counted with multiplicity. The multiplicity of a secant ℓ is computed as $\binom{\#(\ell \cap S)}{2}$. The following upper bound on the smallest size $s_\mu(2, q)$ of a $(1, \mu)$ -saturating set in Π_q is proved:

$$s_\mu(2, q) \leq 2(\mu + 1)\sqrt{(q + 1)\ln(q + 1)} + 2 \sim 2(\mu + 1)\sqrt{q \ln q} \text{ for } 2 \leq \mu \leq \sqrt{q}.$$

By using inductive constructions, upper bounds on the smallest size of a saturating set (as well as on a $(1, \mu)$ -saturating set) in the projective space $PG(N, q)$ are obtained.

All the results are also stated in terms of linear covering codes.

1 Introduction

We denote by Π_q a projective plane (not necessarily Desarguesian) of order q and by $PG(2, q)$ the projective plane over the Galois field with q elements.

Definition 1.1. A point set $S \subset \Pi_q$ is *saturating* if any point of $\Pi_q \setminus S$ is collinear with two points in S .

Saturating sets are considered, for example, in [3, 4, 6, 8–16, 18, 20, 22, 24]; see also the references therein. It should be noted that saturating sets are also called “saturated sets” [8, 18, 20, 24], “spanning sets” [6], “dense sets” [4, 12–14, 16], and “1-saturating sets” [1, 2, 9–11].

The homogeneous coordinates of the points of a saturating set of size k in $PG(2, q)$ form a parity check matrix of a q -ary linear code with length k , codimension 3, and covering radius 2. For an introduction to covering codes see [5, 7]. An online bibliography on covering codes is given in [21].

The main problem in this context is to find small saturating sets (i.e. short covering codes). Let $s(2, q)$ denote the smallest size of a saturating set in Π_q . In [4], by using the probabilistic approach previously introduced in [20], the following upper bound is proved:

$$s(2, q) < 3\sqrt{2}\sqrt{q \ln q} < 5\sqrt{q \ln q}. \quad (1.1)$$

Surveys on random constructions for geometrical objects can be found in [4, 13, 19, 20]; see also the references therein. Saturating sets in $PG(2, q)$ obtained by algebraic constructions or computer search can be found in [3, 6, 8–12, 14–16, 22, 24].

In this paper, we use probabilistic methods to obtain new upper bounds on $s(2, q)$. Our main results are Theorems 1.2 and 1.3 below.

Theorem 1.2. *For the smallest size $s(2, q)$ of a saturating set in a projective plane of order q the following upper bound holds:*

$$s(2, q) \leq 2\sqrt{(q+1)\ln(q+1)} + 2 \sim 2\sqrt{q\ln q}. \quad (1.2)$$

Theorem 1.3. *Let c be a real number greater than or equal to 1 and let k be an integer such that*

$$2c\sqrt{(q+1)\ln(q+1)} + 2 \leq k < \frac{q^2 - 1}{q + 2} \sim q.$$

Then in a projective plane of order q a random point set of size k is a saturating set with probability grater than

$$1 - \frac{1}{(q+1)^{2c^2-2}}. \quad (1.3)$$

Theorem 1.2 improves the constant term of (1.1). It should be noted that our approach is different from those in [4, 20], where random sets lying on two or three lines are considered; in this paper arbitrary random sets are dealt with.

Theorem 1.2 can be expressed in terms of *covering codes*. The *length function* $\ell(R, r, q)$ denotes the smallest length of a q -ary linear code with covering radius R and codimension r ; see [5–7]. Theorem 1.2 can be read as follows.

Corollary 1.4. *The following upper bound on the length function holds.*

$$\ell(2, 3, q) \leq 2\sqrt{(q+1)\ln(q+1)} + 2 \sim 2\sqrt{q\ln q}.$$

Our probabilistic approach can also be applied to *multiple saturating* sets.

Definition 1.5. A point set $S \subset \Pi_q$ is $(1, \mu)$ -*saturating* if for every point Q of $\Pi_q \setminus S$ the number of secants of S through Q is at least μ , counted with multiplicity. Here the multiplicity of a secant ℓ is computed as $\binom{\#(\ell \cap S)}{2}$.

For $\mu = 1$, a $(1, \mu)$ -saturating set is a saturating set as in Definition 1.1.

The homogeneous coordinates of the points of a $(1, \mu)$ -saturating set of size k in $PG(2, q)$ form a parity check matrix of a q -ary linear code with length k , codimension 3, covering radius 2. Such a code is a $(2, \mu)$ -multiple covering of the farthest-off-points $((2, \mu)$ -MCF code or simply MCF code, for short). For an introduction to multiple saturating sets and MCF codes see [7, Chapters 13, 14], [1, 2, 15, 17, 23], and references therein.

The main problem in this context is to find small $(1, \mu)$ -saturating sets (i.e. short MCF codes). Let $s_\mu(2, q)$ be the smallest size of a $(1, \mu)$ -saturating set in Π_q . Our main results on $(1, \mu)$ -saturating sets in Π_q are the following.

Theorem 1.6. *Let $\mu \geq 2$. For the smallest size $s_\mu(2, q)$ of a $(1, \mu)$ -saturating set in a projective plane of order q the following upper bounds hold.*

$$s_\mu(2, q) \leq 2D_\mu \sqrt{(q+1) \ln(q+1)} + 2 \sim 2D_\mu \sqrt{q \ln q}, \quad (1.4)$$

where

$$D_\mu \leq \begin{cases} 2.4 & \text{for } \mu = 2, & q \geq 97 \\ 2.6 & \text{for } \mu = 3, & q \geq 181 \\ 2.8 & \text{for } \mu = 4, & q \geq 125 \\ \mu + 1 & \text{for } \mu \leq \sqrt{q}, & q \geq 4 \\ 2\mu - 1 & \text{for } \mu \leq \frac{1}{2}((1 - \delta)q - \delta + 1) + 1, & q \geq 3 \end{cases}, \quad (1.5)$$

$$\delta = \frac{1}{\sqrt{(q+1) \ln(q+1)}}. \quad (1.6)$$

The μ -length function $\ell_\mu(R, r, q)$ denotes the smallest length of a linear q -ary (R, μ) -MCF code with covering radius R and codimension r [1, 2, 15, 23]. For $\mu = 1$, $\ell_1(R, r, q)$ is the usual length function $\ell(R, r, q)$ for 1-fold coverings. In the *covering code language*, Theorem 1.6 can be read as follows.

Corollary 1.7. *Let D_μ be as in (1.5). The following upper bound on the μ -length function holds.*

$$\ell_\mu(2, 3, q) \leq 2D_\mu \sqrt{(q+1) \ln(q+1)} + 2 \sim 2D_\mu \sqrt{q \ln q}. \quad (1.7)$$

In [1, Prop. 5.2] the following upper bounds on the μ -length function were obtained by adapting the probabilistic approach in [4, 20]:

$$s_\mu(2, q) \leq \ell_\mu(2, 3, q) < 66\sqrt{\mu q \ln q} \text{ for } \mu < 121q \ln q. \quad (1.8)$$

The bounds (1.4) and (1.7) improve (1.8) provided that $D_\mu < 33\sqrt{\mu}$. This actually happens for a wide region of μ , see (1.5).

Let $PG(N, q)$ be the N -dimensional projective space over the Galois field of q elements.

From (1.2) and (1.4), by using inductive constructions from [1, 8, 9], upper bounds on the smallest size of a saturating set in the N -dimensional projective space $PG(N, q)$ can be obtained; see Section 5. In many cases these bounds are better than the known ones.

The paper is organized as follows. In Section 2, we deal with upper bounds on the smallest size of a saturating set in a projective plane; Theorems 1.2 and 1.3 are proved using probabilistic methods. In Sections 3 and 4 we apply our probabilistic approach to $(1, \mu)$ -saturating sets in a projective plane. Finally, in Section 5, bounds for saturating and $(1, \mu)$ -saturating sets in the projective space $PG(N, q)$ are obtained.

2 Upper bound on the smallest size of a saturating set in a projective plane

Let $w > 0$ be a fixed integer. Consider a random $(w + 1)$ -point subset \mathcal{K}_{w+1} of Π_q . The total number of such subsets is $\binom{q^2+q+1}{w+1}$. A fixed point A of Π_q is *covered* by \mathcal{K}_{w+1} if it belongs to an r -secant of \mathcal{K}_{w+1} with $r \geq 2$. We denote by $\text{Prob}(\diamond)$ the probability of some event \diamond .

We estimate

$$\pi := \text{Prob}(A \text{ not covered by } \mathcal{K}_{w+1})$$

as the ratio of the number of $(w + 1)$ -point subsets not covering A over the total number of subsets of size $(w + 1)$. Since a set \mathcal{K}_{w+1} does not cover A if and only if every line through A contains at most one point of \mathcal{K}_{w+1} , we have

$$\pi = \frac{q^{w+1} \binom{q+1}{w+1}}{\binom{q^2+q+1}{w+1}}. \quad (2.1)$$

By straightforward calculations,

$$\begin{aligned} \pi &= \frac{(q^2 + q)(q^2) \cdots (q^2 + q - iq) \cdots (q^2 + q - wq)}{(q^2 + q + 1)(q^2 + q) \cdots (q^2 + q - i) \cdots (q^2 + q + 1 - w)} = \\ &= \prod_{i=0}^w \frac{q^2 + q - iq}{q^2 + q + 1 - i} = \prod_{i=0}^w \left(1 - \frac{iq - i + 1}{q^2 + q + 1 - i} \right) < \prod_{i=0}^w \left(1 - \frac{i(q-1)}{q^2 + q + 1} \right). \end{aligned} \quad (2.2)$$

Using the inequality $1 - x \leq e^{-x}$ we obtain that

$$\pi < e^{-\sum_{i=0}^w \frac{i(q-1)}{q^2+q+1}} = e^{-\frac{(w^2+w)(q-1)}{2(q^2+q+1)}},$$

which implies

$$\pi < e^{-\frac{(w^2+w)(q-1)}{2(q^2+q+1)}} < e^{-\frac{w^2}{2q+2}}, \quad (2.3)$$

provided that

$$\frac{(w+1)(q^2-1)}{(q^2+q+1)} > w$$

that is

$$w < \frac{q^2-1}{q+2} \sim q.$$

The set \mathcal{K}_{w+1} is not saturating if at least one point $A \in \Pi_q$ is not covered by \mathcal{K}_{w+1} . Similarly to [4, Proposition 4.1], we note that

$$\text{Prob}(\mathcal{K}_{w+1} \text{ is not saturating}) \leq \sum_{A \in \Pi_q} \text{Prob}(A \text{ is not covered}). \quad (2.4)$$

Now, using (2.3), we obtain that

$$\text{Prob}(\mathcal{K}_{w+1} \text{ is not saturating}) \leq (q^2 + q + 1)\pi < (q + 1)^2 e^{-\frac{w^2}{2q+2}}. \quad (2.5)$$

Therefore, the probability that all the points of Π_q are covered is

$$\text{Prob}(\mathcal{K}_{w+1} \text{ is saturating}) > 1 - (q + 1)^2 e^{-\frac{w^2}{2q+2}}. \quad (2.6)$$

This quantity is larger than 0 taking, for instance,

$$w = \left\lceil \sqrt{(2q + 2) \ln((q + 1)^2)} \right\rceil.$$

This shows that in Π_q there exists a saturating set with size

$$k \leq 2\sqrt{(q + 1) \ln(q + 1)} + 2 \sim 2\sqrt{q \ln q}.$$

and therefore Theorem 1.2 is proved.

In conclusion, we note that any value

$$w = \left\lceil c\sqrt{(2q + 2) \ln((q + 1)^2)} \right\rceil < \frac{q^2 - 1}{q + 2}$$

where the parameter $c \geq 1$ is independent of q , provides in (2.6) a positive probability greater than $1 - 1/(q + 1)^{2c^2 - 2}$; therefore Theorem 1.3 holds.

It is worth noting that in (1.3) for q large enough, choosing $c = 1 + \varepsilon$, with $\varepsilon = o(1) > 0$, the probability is close to 1.

3 Upper bounds on the smallest size of a $(1, \mu)$ -saturating set in a projective plane, $\mu \geq 2$

For $\mu \geq 2$, we construct a $(1, \mu)$ -saturating set \mathcal{S}_μ in Π_q by joining a $(1, \mu - 1)$ -saturating set $\mathcal{S}_{\mu-1}$ and a “usual” saturating set *disjoint* from $\mathcal{S}_{\mu-1}$.

Let $w > 0$ be a fixed integer; we consider a random $(w + 1)$ -point subset \mathcal{H}_{w+1} of Π_q disjoint from $\mathcal{S}_{\mu-1}$. Let k denote the size of $\mathcal{S}_{\mu-1}$. Then the total number of such subsets is $\binom{q^2 + q + 1 - k}{w + 1}$.

Clearly, if \mathcal{H}_{w+1} is a saturating set then $\mathcal{S}_{\mu-1} \cup \mathcal{H}_{w+1}$ is a $(1, \mu)$ -saturating set.

We argue as in Section 2. For a fixed point A of Π_q we estimate

$$\lambda := \text{Prob}(A \text{ not covered by } \mathcal{H}_{w+1})$$

as the ratio of the number of $(w+1)$ -point subsets not covering A and disjoint from $\mathcal{S}_{\mu-1}$ over the total number of subsets of size $(w+1)$ disjoint from $\mathcal{S}_{\mu-1}$. Similarly to (2.1), we have

$$\lambda < \frac{q^{w+1} \binom{q+1}{w+1}}{\binom{q^2+q+1-k}{w+1}}. \quad (3.1)$$

In fact, the number of $(w+1)$ -point subsets not covering A and disjoint from $\mathcal{S}_{\mu-1}$ is smaller than the numerator of (3.1).

By straightforward calculations similar to (2.2),

$$\lambda < \frac{(q^2+q)(q^2) \cdots (q^2+q-wq)}{(q^2+q+1-k)(q^2+q-k) \cdots (q^2+q+1-w-k)} < \prod_{i=0}^w \left(1 - \frac{i(q-1)-k}{q^2+q+1}\right).$$

Now, under the condition $w < \frac{q^2-1}{q+2} \sim q$, as in (2.3) we obtain that

$$\lambda < e^{-\sum_{i=0}^w \frac{i(q-1)-k}{q^2+q+1}} < e^{-\frac{(w^2+w)(q-1)}{2(q^2+q+1)} + \frac{k(w+1)}{2q(q+1)}} < e^{-\frac{w^2}{2q+2} + \frac{k(w+1)}{2q(q+1)}}.$$

This implies that

$$\text{Prob}(\mathcal{H}_{w+1} \text{ is not saturating}) \leq (q^2+q+1)\lambda < (q+1)^2 e^{-\frac{w^2}{2q+2} + \frac{k(w+1)}{2q(q+1)}}.$$

So,

$$\text{Prob}(\mathcal{S}_{\mu-1} \cup \mathcal{H}_{w+1} \text{ is } (1, \mu)\text{-saturating}) > 1 - (q+1)^2 e^{-\frac{w^2}{2q+2} + \frac{k(w+1)}{2q(q+1)}}. \quad (3.2)$$

Throughout this section, δ is as in (1.6). We represent w and k in the following form:

$$w = \left\lceil d\sqrt{(2q+2)\ln((q+1)^2)} \right\rceil, \quad d > 1 \text{ independent of } q; \quad (3.3)$$

$$k \leq 2D\sqrt{(q+1)\ln(q+1)} + 2, \quad D \geq 1 \text{ independent of } q.$$

Then the following holds:

$$w+1 \leq 2(d+\delta)\sqrt{(q+1)\ln(q+1)}; \quad (3.4)$$

$$k \leq 2(D+\delta)\sqrt{(q+1)\ln(q+1)}; \quad (3.5)$$

$$(q+1)^2 e^{-\frac{w^2}{2q+2} + \frac{k(w+1)}{2q(q+1)}} < \frac{(q+1)^{2(D+\delta)(d+\delta)/q}}{(q+1)^{2d^2-2}} = (q+1)^{\frac{2}{d}(D+\delta)(d+\delta)-2(d^2-1)}.$$

Let

$$d = 1 + \frac{D+\delta}{q}. \quad (3.6)$$

Then

$$\begin{aligned} d-1 &= \frac{D+\delta}{q} > \frac{2(D+\delta)(d+\delta)}{2q(d+1)}; \\ \frac{2}{q}(D+\delta)(d+\delta) - 2(d^2-1) &< 0; \\ (q+1)^2 e^{-\frac{w^2}{2q+2} + \frac{k(w+1)}{2q(q+1)}} &< 1. \end{aligned}$$

The last inequality means that the probability in (3.2) is positive. As $\#(\mathcal{S}_{\mu-1} \cup \mathcal{H}_{w+1}) = k + w + 1$, taking into account (1.6), (3.4)–(3.6), we have proved the following lemma.

Lemma 3.1. *Let Π_q be a projective plane of order q . Let $\mu \geq 2$ and assume that for some $D \geq 1$ in Π_q there exists a $(1, \mu-1)$ -saturating set with size $k \leq 2D\sqrt{(q+1)\ln(q+1)} + 2$. Then in Π_q there exists a $(1, \mu)$ -saturating set with size*

$$v \leq 2 \left(D + 1 + \frac{D+\delta}{q} + \delta \right) \sqrt{(q+1)\ln(q+1)} + 2. \quad (3.7)$$

Corollary 3.2. *Let $\mu \geq 2$.*

(i) *In Π_q there is a $(1, \mu)$ -saturating set with size*

$$k \leq 2D_\mu \sqrt{(q+1)\ln(q+1)} + 2,$$

where

$$D_1 = 1, \quad D_i = D_{i-1} + 1 + \frac{D_{i-1} + \delta}{q} + \delta, \quad i = 2, 3, \dots, \mu. \quad (3.8)$$

(ii) *In Π_q there is a $(1, \mu)$ -saturating set with size*

$$k \leq 2(\mu+1)\sqrt{(q+1)\ln(q+1)} + 2, \quad \mu \leq \sqrt{q}.$$

(iii) *In Π_q there is a $(1, \mu)$ -saturating set with size*

$$k \leq 2(2\mu-1)\sqrt{(q+1)\ln(q+1)} + 2$$

provided that

$$\mu \leq \frac{(1-\delta)q - \delta + 1}{2} + 1.$$

Proof. (i) For $\mu = 1$, a $(1, \mu)$ -saturating set is an usual saturating set. Therefore, we may use Theorem 1.2 and put $D_1 = 1$. Then we iteratively apply (3.7).

(ii) We use (i). By (3.8),

$$D_i = i + \frac{\sum_{j=1}^{i-1} D_j + (i-1)\delta}{q} + (i-1)\delta, \quad i = 2, 3, \dots, \mu. \quad (3.9)$$

By induction, we will show that, under the condition $\mu \leq \sqrt{q}$,

$$A_i := \frac{\sum_{j=1}^{i-1} D_j + (i-1)\delta}{q} + (i-1)\delta \leq 1$$

holds for $i = 2, 3, \dots, \mu$. We have

$$A_2 = \frac{D_1 + \delta}{q} + \delta = \frac{1 + \delta}{q} + \delta < 1.$$

Assume that $A_i \leq 1$, $i = 2, 3, \dots, h$, with $h \leq \mu - 1 \leq \sqrt{q} - 1$. Then, by (3.8), (3.9), we have that

$$\begin{aligned} D_1 &= 1; \quad D_i \leq i + 1, \quad i = 2, 3, \dots, h; \\ A_{h+1} &\leq \frac{1 + \sum_{j=2}^h (j+1) + h\delta}{q} + h\delta = \frac{h^2 + 3h - 2 + 2h\delta}{2q} + h\delta. \end{aligned}$$

It can be checked that $A_{h+1} < 1$ if $h \leq \sqrt{q} - 1$. So, $D_i \leq i + 1$, $i = 2, 3, \dots, \mu$.

(iii) By the proof of (ii), $D_2 \leq 3$ holds. Assume that $D_i \leq 2i - 1$, $i = 2, 3, \dots, h$, with $h \leq \mu - 1 \leq \frac{1}{2}((1 - \delta)q - \delta + 1)$. Then

$$\begin{aligned} D_{h+1} &= D_h + 1 + \frac{D_h + \delta}{q} + \delta \leq 2h - 1 + 1 + \frac{2h - 1 + \delta}{q} + \delta \leq \\ &2h + \frac{(1 - \delta)q - \delta + 1 - 1 + \delta}{q} + \delta = 2h + (1 - \delta) + \delta = 2h + 1. \end{aligned}$$

□

4 Improved upper bounds on the smallest size of a $(1, \mu)$ -saturating set in a projective plane, $\mu = 2, 3, 4$

Let $w > 0$ be a fixed integer. We consider a random $(w + 1)$ -point subset \mathcal{K}_{w+1} of Π_q . The total number of such subsets is $\binom{q^2 + q + 1}{w+1}$. As above, let A be a fixed point of Π_q .

We say that \mathcal{K}_{w+1} covers A *exactly* i times if the number of secants of \mathcal{K}_{w+1} through A is *exactly* i , counted with multiplicity. Denote by T_i the number of $(w + 1)$ -subsets

covering A exactly i times, $i = 0, 1, 2, \dots$, where $i = 0$ means that A is not covered by \mathcal{K}_{w+1} . Similarly to the numerator of (2.1) we have

$$T_0 = q^{w+1} \binom{q+1}{w+1}. \quad (4.1)$$

According to Definition 1.5, we say that a fixed point A of Π_q is μ -covered by \mathcal{K}_{w+1} if the number of secants of \mathcal{K}_{w+1} through A is at least μ , counted with multiplicity. We estimate

$$\pi_\mu := \text{Prob}(A \text{ not } \mu\text{-covered by } \mathcal{K}_{w+1})$$

as the ratio of the number of $(w+1)$ -point subsets that do not μ -cover A over the total number of subsets of size $(w+1)$. So,

$$\pi_\mu = \frac{\sum_{i=0}^{\mu-1} T_i}{\binom{q^2+q+1}{w+1}} = R_{w,q} \sum_{i=0}^{\mu-1} \frac{T_i}{T_0} \quad (4.2)$$

where

$$R_{w,q} = \frac{T_0}{\binom{q^2+q+1}{w+1}}. \quad (4.3)$$

The set \mathcal{K}_{w+1} is not $(1, \mu)$ -saturating if at least one point $A \in \Pi_q$ is not μ -covered by \mathcal{K}_{w+1} . As in (2.4) and (2.5), we have

$$\text{Prob}(\mathcal{K}_{w+1} \text{ is not } (1, \mu)\text{-saturating}) \leq (q^2 + q + 1)\pi_\mu < (q + 1)^2\pi_\mu.$$

Hence, the probability that all the points of Π_q are μ -covered is

$$\text{Prob}(\mathcal{K}_{w+1} \text{ is a } (1, \mu)\text{-saturating}) \geq 1 - (q^2 + q + 1)\pi_\mu > 1 - (q + 1)^2\pi_\mu. \quad (4.4)$$

Throughout this section, we represent w in the form (3.3). Also, from now, we assume

$$w < \frac{q+1}{2}. \quad (4.5)$$

Theorem 4.1. *For the smallest size $s_\mu(2, q)$ of a $(1, \mu)$ -saturating set in a projective plane of order q the following upper bounds hold:*

$$s_2(2, q) \leq 2.4\sqrt{(q+1)\ln(q+1)} + 2 \sim 2.4\sqrt{q\ln q}, \quad q \geq 97; \quad (4.6)$$

$$s_3(2, q) \leq 2.6\sqrt{(q+1)\ln(q+1)} + 2 \sim 2.6\sqrt{q\ln q}, \quad q \geq 181; \quad (4.7)$$

$$s_4(2, q) \leq 2.8\sqrt{(q+1)\ln(q+1)} + 2 \sim 2.8\sqrt{q\ln q}, \quad q \geq 125. \quad (4.8)$$

Proof. Let

$$\widehat{w} = d\sqrt{(2q+2)\ln((q+1)^2)}. \quad (4.9)$$

We first establish some inequalities that will be useful in the proof below. From (2.1), (2.3), (4.1), (4.3), (4.5), and (4.9), it is easy to see that

$$q+1-2w > 0, \quad 2(q+f-w) > q+1 \text{ if } f \geq 1, \quad \widehat{w} \leq w < \widehat{w}+1, \quad (4.10)$$

$$R_{w,q} < e^{-\frac{w^2}{2q+2}} \leq e^{-\frac{\widehat{w}^2}{2q+2}}, \quad w^2 \pm w < 2\widehat{w}^2, \quad (w-2)(w-1)w(w+1) < 2\widehat{w}^4, \\ (w-1)w(w+1) < 3\widehat{w}^3, \quad (w-4)(w-3)(w-2)(w-1)w(w+1) < \widehat{w}^6.$$

A set \mathcal{K}_{w+1} covers A exactly once if one line through A contains two points of \mathcal{K}_{w+1} , whereas each of the remaining q lines contains at most one point of \mathcal{K}_{w+1} . So,

$$T_1 = (q+1) \binom{q}{2} \cdot q^{w-1} \binom{q}{w-1}. \quad (4.11)$$

A set \mathcal{K}_{w+1} covers A exactly twice if some two lines through A contains two points of \mathcal{K}_{w+1} , whereas each of the remaining $q-1$ lines contains at most one point of \mathcal{K}_{w+1} . So,

$$T_2 = \binom{q+1}{2} \binom{q}{2}^2 \cdot q^{w-3} \binom{q-1}{w-3}. \quad (4.12)$$

Finally, a set \mathcal{K}_{w+1} covers A exactly 3 times in the following two cases:

- one line through A contains three points of \mathcal{K}_{w+1} , whereas each of the remaining q lines contains at most one point of \mathcal{K}_{w+1} ;
- three lines through A contain two points of \mathcal{K}_{w+1} , whereas each of the remaining $q-2$ lines contains at most one point of \mathcal{K}_{w+1} .

Therefore,

$$T_3 = (q+1) \binom{q}{3} \cdot q^{w-2} \binom{q}{w-2} + \binom{q+1}{3} \binom{q}{2}^3 \cdot q^{w-5} \binom{q-2}{w-5}. \quad (4.13)$$

Let $\mu = 2$. Taking into account (3.3), (4.1), (4.2), (4.9) – (4.11), we have

$$\pi_2 = R_{w,q} \left(1 + \frac{T_1}{T_0}\right) = R_{w,q} \left(1 + \frac{w(w+1)(q-1)}{2q(q+1-w)}\right) < e^{-\frac{w^2}{2q+2}} \left(1 + \frac{w(w+1)}{2(q+1-w)}\right) = \\ \frac{2q+2+w^2-w}{2(q+1-w)} e^{-\frac{w^2}{2q+2}} < \frac{2q+2+2\widehat{w}^2}{q+1} e^{-\frac{\widehat{w}^2}{2q+2}} = \frac{2+8d^2\ln(q+1)}{(q+1)^{2d^2}}.$$

By a computer aided computation,

$$(q+1)^2\pi_2 = \frac{2+8d^2\ln(q+1)}{(q+1)^{2d^2-2}} < 1 \text{ if } d = 1.2, \quad q \geq 97. \quad (4.14)$$

Under condition (4.14), the probability in (4.4) is positive. So, taking into account (3.3), the upper bound in (4.6) is proved.

Let $\mu = 3$. Taking into account (3.3), (4.1), (4.2), (4.9) – (4.12), we have

$$\begin{aligned}\pi_3 &= R_{w,q} \left(1 + \frac{w(w+1)(q-1)}{2q(q+1-w)} + \frac{(w-2)(w-1)w(w+1)(q-1)^2}{8q^2(q+2-w)(q+1-w)} \right) < \\ e^{-\frac{w^2}{2q+2}} \left(1 + \frac{w(w+1)}{2(q+1-w)} + \frac{(w-2)(w-1)w(w+1)}{8(q+2-w)(q+1-w)} \right) < \\ e^{-\frac{\widehat{w}^2}{2q+2}} \left(1 + \frac{2\widehat{w}^2}{q+1} + \frac{2\widehat{w}^4}{2(q+1)^2} \right) &= \frac{1 + 8d^2 \ln(q+1) + 16d^4 \ln^2(q+1)}{(q+1)^{2d^2}}.\end{aligned}$$

By a computer aided computation,

$$(q+1)^2 \pi_3 = \frac{1 + 8d^2 \ln(q+1) + 16d^4 \ln^2(q+1)}{(q+1)^{2d^2-2}} < 1 \text{ if } d = 1.3, \ q \geq 181. \quad (4.15)$$

Under condition (4.15), the probability in (4.4) is positive. So, taking into account (3.3), the upper bound in (4.7) is proved.

Finally, let $\mu = 4$. Taking into account (3.3), (4.1), (4.2), (4.9) – (4.13), we have

$$\begin{aligned}\pi_4 &= \pi_3 + R_{w,q} \frac{T_3}{T_0} = \pi_3 + R_{w,q} \left(\frac{(w-1)w(w+1)(q-1)(q-2)}{6q^2(q+2-w)(q+1-w)} + \right. \\ &\quad \left. \frac{(w-4)(w-3)(w-2)(w-1)w(w+1)(q-1)^3}{48q^3(q+3-w)(q+2-w)(q+1-w)} \right) < \pi_3 + e^{-\frac{\widehat{w}^2}{2q+2}} \left(\frac{3\widehat{w}^3}{(q+1)^2} + \frac{\widehat{w}^6}{6(q+1)^3} \right).\end{aligned}$$

It can be checked by computer that

$$(q+1)^2 \pi_4 < 1 \text{ if } d = 1.4, \ q \geq 125. \quad (4.16)$$

Under condition (4.16), the probability in (4.4) is positive. So, taking into account (3.3), also (4.8) is proved. \square

5 Upper bounds on the smallest size of a saturating set in the projective space $PG(N, q)$

A point set $S \subset PG(N, q)$ is *saturating* if any point of $PG(N, q) \setminus S$ is collinear with two points in S . Results on saturating sets in $PG(N, q)$ can be found for instance in [6, 8–10, 15, 18, 24] and the references therein.

Let $[n, n-r]_q R$ be a linear q -ary code of length n , codimension r , and covering radius R . The homogeneous coordinates of the points of a saturating set with size n in $PG(r-1, q)$,

form a parity check matrix of an $[n, n - r]_q 2$ code; see [6–9, 15, 18]. Let $s(N, q)$ be the smallest size of a saturating set in $PG(N, q)$, $N \geq 3$. In terms of covering codes, we recall the equality $s(N, q) = \ell(2, N + 1, q)$.

Proposition 5.1. *For the smallest size $s(N, q)$ of a saturating set in the projective space $PG(N, q)$ and for the length function $\ell(2, N + 1, q)$, the following upper bound holds:*

$$s(N, q) = \ell(2, N + 1, q) \leq \left(2\sqrt{(q+1)\ln(q+1)} + 2\right) q^{\frac{N-2}{2}} + 2q^{\frac{N-4}{2}} \sim 2q^{\frac{N-1}{2}} \sqrt{\ln q}, \quad N = 2t - 2 \geq 6, \quad (5.1)$$

where $t = 4, 6$ and $t \geq 8$, $N \neq 8, 12$, $q \geq 79$.

Proof. By Theorem 1.2, there is a saturating set with size $n_q = 2\sqrt{(q+1)\ln(q+1)} + 2$ in $PG(2, q)$. From the corresponding $[n_q, n_q - 3]_q 2$ code, by using the construction of [8, Ex. 6], see also [9, Th. 4.4], one can obtain an $[n, n - r]_q 2$ code with $r = 2t - 1 \geq 7$, $r \neq 9, 13$, $n = n_q q^{t-2} + 2q^{t-3}$, under condition $q + 1 \geq 2n_q$ that holds for $q \geq 79$. \square

Surveys of the known $[n, n - r]_q 2$ codes and saturating sets in $PG(N, q)$ can be found in [8, 9, 15]. In many cases bound (5.1) is better than the known ones.

A point set $S \subset PG(N, q)$ is $(1, \mu)$ -saturating if for every point Q of $PG(N, q) \setminus S$ the number of secants of S through Q is at least μ , counted with multiplicity. The multiplicity of a secant ℓ is computed as $\binom{\#(\ell \cap S)}{2}$ [1, 2].

Let $[n, n - r]_q(R, \mu)$ be a linear q -ary (R, μ) -MCF code of length n , codimension r , and covering radius R . The points of a $(1, \mu)$ -saturating set with size n in $PG(r - 1, q)$ form a parity check matrix of an $[n, n - r]_q(2, \mu)$ code; see [1, 15, 23]. Let $s_\mu(N, q)$ be the smallest size of a $(1, \mu)$ -saturating set in $PG(N, q)$, $N \geq 3$.

Proposition 5.2. *For the smallest size $s_\mu(N, q)$ of a $(1, \mu)$ -saturating set in the projective space $PG(N, q)$, $N \geq 4$ even, and for the μ -length function, the following upper bound holds:*

$$s_\mu(N, q) = \ell_\mu(2, N + 1, q) \leq q^{\frac{N-2}{2}} n_{q,\mu} + \max(3, \mu) \frac{q^{\frac{N-2}{2}} - 1}{q - 1} \sim 2D_\mu q^{\frac{N-1}{2}} \sqrt{\ln q}, \quad (5.2)$$

where $n_{q,\mu} = 2D_\mu \sqrt{(q+1)\ln(q+1)} + 2$, D_μ is as in (1.5), $q^{\frac{N-2}{2}} + 1 - \mu \geq n_{q,\mu}$.

Proof. By Theorem 1.6, there exists a $(1, \mu)$ -saturating set with size $n_{q,\mu}$ in $PG(2, q)$. We directly apply [1, Cor. 6.5] to the corresponding $[n_{q,\mu}, n_{q,\mu} - 3]_q(2, \mu)$ MCF code and use the one-to-one correspondence between $(1, \mu)$ -saturating sets and $(2, \mu)$ -MCF codes. \square

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